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Presentations of semigroups and embeddings in inverse semigroups

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Let X be a finite set of alphabets, X^* the free monoid generated by X and R a finite set of $X^* \times X^*$. Then let $(X; R)$ denote the factor semigroup of X^* modulo the congruence generated by the relation R. Then we say that a semigroup S has a representation $(X; R)$ if S is isomorphic to $(X; R)$. In this paper, we study relation R for which $S = (X; R)$ can be embedded in an inverse semigroup.

Stephen [2] gave a method of studying word problems for inverse semigroups with a presentation in terms of inverse word graphs.

We shall apply this method to invesetigate embeddability of semigroups with a presentation into an inverse semigroup.

1 Embedding theorem

We recall from Stephen [2] theory of inverse words graphs to give an embedding theorem.

A ***labeled digraph***, Γ, over a non-empty set T consists of a set vertices, $V(\Gamma)$, and a set of edges, $E(\Gamma)$, where $E(\Gamma) \subseteq V(\Gamma)\times\ T\ \times V(\Gamma)$. An edge (v_1, x, v_2) is said to be ***labeled*** by x and ***directed*** from v_1 to v_2. Then vertex v_1 is said to be yhe ***initial*** vertex of the edge and the vertex v_2 is said to be the ***terminal*** vertex of the edge. A (directed) ***path*** is a sequence of edges such that the terminal vertex of one edge is the initial vertex of the edge. A path which starts and ends at the same vertex is said to be a ***loop*** at that vertex. Teh graph Γ is said to be ***strongly connected*** if given any two vertices, α, β, there is a directed path p from α, β. We will also call p a path from α, β ***walk***. For a path p, $W(p) \in T^*$ is denotes the word that labels the path. If $w = W(p)$ labels the α, β walk p, then we will sometimes denote the endpoint β of p by αw. The labeled digraph Γ is said to be ***finite*** if both $E(\Gamma)$ and $V(\Gamma)$ are finite sets. A labeled digraph is called ***deterministic*** if all edges directed from a vertex are labeled by different letters, and ***injective*** if all edges towards a vertex are labeled by different letters.

For any $w \in T^*$, $|w|$ is the number of elements of T, including repetitions, occurring in w. Let X be a non-empty set. Then we put $X^{-1} = \{x^{-1}|x \in X\}$, where X and X^{-1} are disjoint.

Definition 1. An ***inverse word graph over*** $X \cup X^{-1}$ is a strongly connected digraph, Γ, with edges labeled from $X \cup X^{-1}$ in such a way that the labeling is consistent with an involutions; that is, (γ, y, δ) is an edge in Γ if and only if (δ, y^{-1}, γ) is an edge in Γ, where $y \in X \cup X^{-1}$.

We note that in inverse word graph, "deterministic" is equivalent to "injective". We will also assume that at each vertex there is an empty path, which is a loop, labeled by 1.

Definition 2. A ***birooted inverse word graph*** (over $X \cup X^{-1}$) is a triple $A = (\alpha, \Gamma, \beta)$, where Γ is an inverse word graph over $X \cup X^{-1}$, and α and β are distinguished vertices of Γ called, respectively, ***start*** and ***end*** of A

Definition 3. Let Γ be a inverse word graph and η be an equivalence relation, called V-***equivalence***, on the set of vertices of Γ. Then V-equivalence induces a new inverse word graph as following;
V-***quotient*** of Γ by η is the labeled digraph Γ/η consisting of $V(\Gamma)/\eta$ and $E(\Gamma/\eta)=$ $\{((v_1\eta), x, (v_2\eta)) \ |(v_1, x, v_2) \in E(\Gamma)\}$

A V-equivalence, η, on Γ, also induces a V-quotient on a birooted inverse word graph $A = (\alpha, \Gamma, \beta)$, that is, $A/\eta = ((\alpha\eta), \Gamma/\eta, (\beta\eta))$.

The remain argument in this section, $P = (X; S)$ will be a fixed presentation of an inverse monoid $M = Inv < X|S >= (X \cup X^{-1})^*/\tau$, τ the corresponding congruence on $(X \cup X^{-1})^*$.

Definition 4. For a word $w \in (X \cup X^{-1})^*$, where $w = w_1 w_2 \cdots w_n$, ***the linear graph of*** w is the birooted inverse word graph $(\alpha_w, \Gamma_w, \beta_w)$, where $V(\Gamma_w) = \{\alpha_1\alpha_2 \cdots, \alpha_{n+1}\}$ and $E(\Gamma_w) = \{(\alpha_1, w_1, \alpha_2), (\alpha_2, w_2, \alpha_3), \cdots, (\alpha_n, w_n, \alpha_{n+1})\}$.

Next, we shall introduce two constructions on birooted inverse word graph which produces a new graph.

Definition 5. Let (α, Γ, β) be an inverse word graph over $(X \cup X^{-1})^*$. If Γ has two directed edges with a common initial vertex δ and the same label (δ, y, γ_1) and (δ, y, γ_1) for some $y \in X \cup X^{-1}$, then we form a new birooted inverse word graph by taking the quotient of (α, Γ, β) by the equivalence relation on $V(\Gamma)$ generated by $\{(\gamma_1, \gamma_2)\}$. The resulting birooted inverse word graph has one fewer vertex than (α, Γ, β), and two (or more) fewer edges than (α, Γ, β), and is said to ***be obtained from*** (α, Γ, β) ***by a determination.***

For the presentation, $P = (X; S)$, we give a method of enlarging a birooted inverse word graph.

Definition 6. Let (α, Γ, β) be an inverse word graph over $(X \cup X^{-1})^*$. If $r = s$ is a relation in S(i.e.,(r, s) or $(s, r) \in S$), and Γ has a $\gamma_1 - \gamma_2$ walk labeled by r but no $\gamma_1 - \gamma_2$ walk labeled by s, then we obtain a new birooted inverse word graph $(\alpha', \Gamma', \beta')$ by sewing on the linear graph of s onto (α, Γ, β), by identifying the start and end of $(\alpha_s, \Gamma_s, \beta_s)$ with γ_1 and γ_2, respectively. The start and end of $(\alpha', \Gamma', \beta')$ correspond to the start and end of (α, Γ, β). If $|s| = m$, then $(\alpha', \Gamma', \beta')$ has $m - 1$ new vertices and $2m$ new edges, and is said to ***be obtained from*** (α, Γ, β) ***by an elementary*** P***-expansion.***

A deterministic birooted inverse word graph (α, Γ, β) will be *closed* if no elementary P-expansion is defined on (α, Γ, β).

Definition 7. Let $(X; R)$ be a presentation and w be a word in $(X \cup X^{-1})^*$ and (α, Γ, β) be a linear graph of w. If $(\alpha', \Gamma', \beta')$ is obtained from (α, Γ, β) by a sequence of elementary P-expansions and determinations, and it is closed. Then we say that $(\alpha', \Gamma', \beta')$ is *Schützenberger graph* Γ of w.

Definition 8. Let (α, Γ, β) be a deterministic birooted inverse word graph and a be an element of $(X \cup X^{-1})$. Then we defind a partial one-to-one mapping ϕ_a as follows :

$$\gamma\phi_a = \begin{cases} \delta & \text{if } (\gamma, a, \delta) \in E(\Gamma) \\ \\ \text{undefinded} & \text{otherwise} \end{cases}$$

Moreover, we defind an inverse semigroup generated by partial one-to-one mappings $\{\phi_a\}$, and we call it the *inverse transition semigroup of* (α, Γ, β).

Theorem 1. *Let* $S = < X; R >$ *be a presentation of* S*. Then* S *can be embedded in an inverse semigroup if and only if for each pair of words* s, t *over* X *with* $(s, t) \notin R^*$*, the Schützenberger graphs of* s, t *are distinct from each other.*

In this case S *is embedded in the direct product of inverse transition semigroups on Schützenberger graphs of words, which are reprentatives of the* R^**-classes.*

2 Applications

Actually, in application of Theorem 1 to a presentation $P = (X; R)$, we demand that

(1) the word problem for $(X; R)$ is decidable, (2) it is possible to calculate Schützenberger graphs.

This is a description of proccedure of deciding whether or not a semigroup with a presentation is embedded in an inverse semigroup :

Proccedure. Given a presentation $P = (X; R)$ satisfying the above (1) and (2), where X and R are finite.

Give length and lexicographic ordering on X^*.

(i) choose the minimal word w form a congruence class $[w] = \{w' | (w, W') \in R^*\}$.
(We say that the minimal word w is the canonical form of w' $(w' \in [W])$).
(ii) Depict the linear graph of w.
(iii) Perform to elementary P-expansions and then determinations to the the linear graph. It is the Schützenberger graph if it closed.
(iv) Check whether or not this Schützenberger graph coincides the all Schützenberger graphs obtained before.
(v) If one find two Schützenberger graphs which coincide then this semigroup is not embeddable into a inverse semigroup.

Then end.

(vi) If one does not find two Schützenberger graphs which coincide then we repeat above (i), (ii), (iii), (iV).

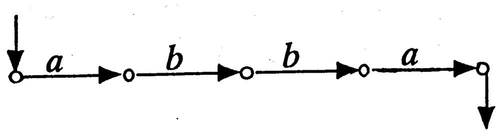
And if we can not find two Schützenberger graphs which coincide for all canonical forms then this semigroup is embeddable into the semigroup $\Pi_{w \in Can(S)} S(w)$ where $Can(S)$ is the set of canonical forms of S.

Here, we shall introduce typical examples which are complete and embeddable, and, complete and unembeddable.

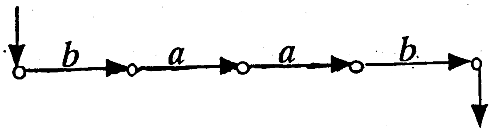
Example 1. $S =< a, b; aba = a >$ is not embeddable into in any inverse semigroup of inverse semigroup.

proof Note that both *abba* and *baab* are the canonical forms. So $[abba]$, $[baab]$ are different from each other in S. However, we shall see that the Schützenberger graphs of *abba*, *baab* are coincide.

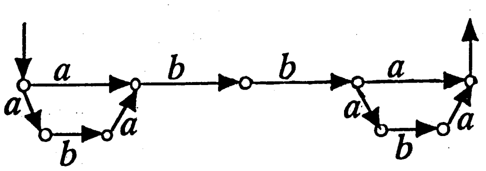
The linear graph of *abba* :

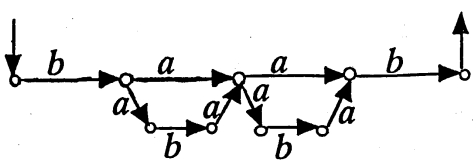


The linear graph of *baab* :



Elementary P-expansions :





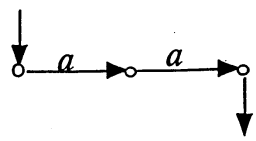
Determinations :

These are the Schützenberger graphs .

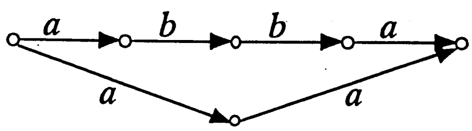
Example 2. $S =< a, b; ab^n a = a^2 >$ is embeddable into the quotient inverse semigroup of inverse semigroup generated by $\{a, b\}$ by the congruence generated by the relation $\{(ab^n a, a^2)\}$.

proof We shall show the case $n = 2$ by the typical example and general case is similar.

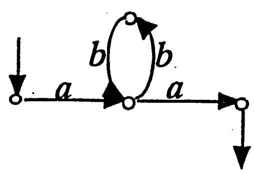
The linear graph of aa :



Elementary P-expansions :



Determinations :



This is the Schützenberger graph .

all Schützenberger graphs of words are obtained by exchanging all linear graphs $\circ\xrightarrow{aa}\circ$ for the graph above. Therefore, this semigroup is embeddable in an inverse semigroup.

Problems. Find types of relations which present semigroups embeddable in an inverse semigroup.

Remark. By using Adjian's result in [1], We can show that if a presentation $(X; R)$ has neither left cycle nor right cycle then the semigroup presented by $(X; R)$ can be embedded in an inverse semigroup.

References

[1] S.I. Adjian. *Defining relations and algorithmic problems for groups ,and semigroups.* Trudy MNath. Inst. Steklov **152**(1967).

[2] J.B. Stephen. *Presentations of inverse monoids.* Journal of Pure and applied Algebra **63**(1990), 81-112.